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Monte Carlo simulations of anomalous diffusion of cosmic rays

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Abstract. The anomalous diffusion model for cosmic rays transport in fractal-like galactic medium developed by the authors allows to describe main features of the observed spectrum of charged particles arriving in the Solar system. Anomalous diffusion equation underlying the model contains fractional derivative operators and do not take into account the finite speed of particles transport. In this regard, we conducted a Monte Carlo simulation of uncoupled continuous-time random walks of particles in this model. The particles can perform an anomalously large jumps between scattering centers (Levy flights) and also can stay a long time in the inhomogeneities of the medium (Levy traps). The probability of such behavior of particles is described by power-law distributions for the jumps length and for the residence time in the inhomogeneity. In this paper, we present the spatial distribution of particles obtained from the direct simulations of particle trajectories in the framework of our model for two versions: considering finite particle speed and the case of an instantaneous jump.

Introduction

Continuous-time random walks (CTRWs) and fractional diffusion equations have received increasing attention.

In our papers [1, 2, 3] we proposed an anomalous diffusion model for solution of the "knee" problem in primary cosmic-rays spectrum and explanation of different values of spectral exponent of protons and other nuclei. The anomaly results from large free paths ("Lévy flights") of particles between magnetic domains. It is also assumed that the particle can spend anomalously a long time in the trap. Thus the walk process in this model can be represented by a model CTRWs.

We use a straightforward Monte Carlo method for the efficient simulation of uncoupled CTRWs and apply it to compute approximate solutions of the Cauchy problem for a anomalous diffusion (AD) equation of cosmic rays that has fractional space and time derivatives. Also, we wanted to analyze the effect of the finite velocity of the particles on the shape of the spectrum. It is important to evaluate the effect because the equation of anomalous diffusion assumes an infinite speed of the jump between the inhomogeneities.

1. Continuous-time random walks

A CTRW [4] is a pure jump process; it consists of a sequence of independent identically distributed (i.i.d.) random jumps (events) ξ_i separated by i.i.d. random waiting times τ_i,

$$t_n = \sum_{i=1}^{n} \tau_i \,, \quad \tau_i \in \underline{\mathrm{R}}_+, \tag{1}$$

so that the position at time $t \in [t_n, t_{n+1})$ is given by

$$x(t) = \sum_{i=1}^{n} \xi_i, \quad \xi_i \in \underline{\mathrm{R}}. \tag{2}$$

A realization of the process is a piecewise constant function resulting from a sequence of up or down steps with different height and depth.

1.1. Symmetric Lévy α-stable probability distribution

The symmetric Lévy α-stable probability density $L_\alpha(\xi)$ for the jumps can be calculated by [5]:

$$\xi_\alpha = \gamma_x \left(\frac{-\log u \cos\phi}{\cos((1-\alpha)\phi)} \right)^{1-1/\alpha} \frac{\sin(\alpha\phi)}{\cos\phi}, \tag{3}$$

where $\phi = \pi(v - 1/2)$, $u, v \in (0, 1)$ are independent uniform random numbers, γ_x is the scale parameter, and ξ_α is a symmetric Lévy α-stable random number. For $\alpha = 2$ reduces to $\xi_2 = 2\gamma_x\sqrt{-\log u}\sin\phi$, i.e. the method for Gaussian deviates. The other two notable limit cases are the Cauchy distribution, with $\alpha = 1$ and $\xi_1 = \gamma_x \tan\phi$, and the Lévy distribution, with $\alpha = 1/2$ and $\xi_{1/2} = -\gamma_x \tan\phi/(2\log u \cos\phi)$.

1.2. One-parameter Mittag-Leffler probability distribution

The probability density $\psi_\beta(\tau)$ for the waiting times can be computed by [5]:

$$\tau_\beta = -\gamma_t \log u \left(\frac{\sin(\beta\pi)}{\tan(\beta\pi v)} - \cos(\beta\pi) \right)^{1/\beta}, \tag{4}$$

where $u, v \in (0, 1)$ are independent uniform random numbers, γ_t is the scale parameter, and τ_β is a Mittag-Leffler random number.

2. Anomalous diffusion equation

AD-propagator $P(\vec{r}, t)$, describing time and spatial distribution of particles, obeys the equation [2]

$$\frac{\partial P}{\partial t} = -D(\alpha, \beta) \mathrm{D}_{0+}^{1-\beta} (-\Delta)^{\alpha/2} P(\vec{r}, t) + \delta(\vec{r})\delta(t). \tag{5}$$

Here, D is the AD coefficient, D_{0+}^{μ} denotes the Riemann-Liouville fractional derivative [6], $(-\Delta)^{\alpha/2}$ is the fractional Laplacian (called "Riss" operator) [6], $\vec{r} \in \mathbf{R}^m$.

In the case of punctual instantaneous source cosmic ray distribution is

$$P(\vec{r}, t) = \left(D(R, \alpha, \beta) t^\beta \right)^{-3/\alpha} \Psi_m^{(\alpha,\beta)} \left(r (D(\alpha, \beta) t^\beta)^{-1/\alpha} \right), \tag{6}$$

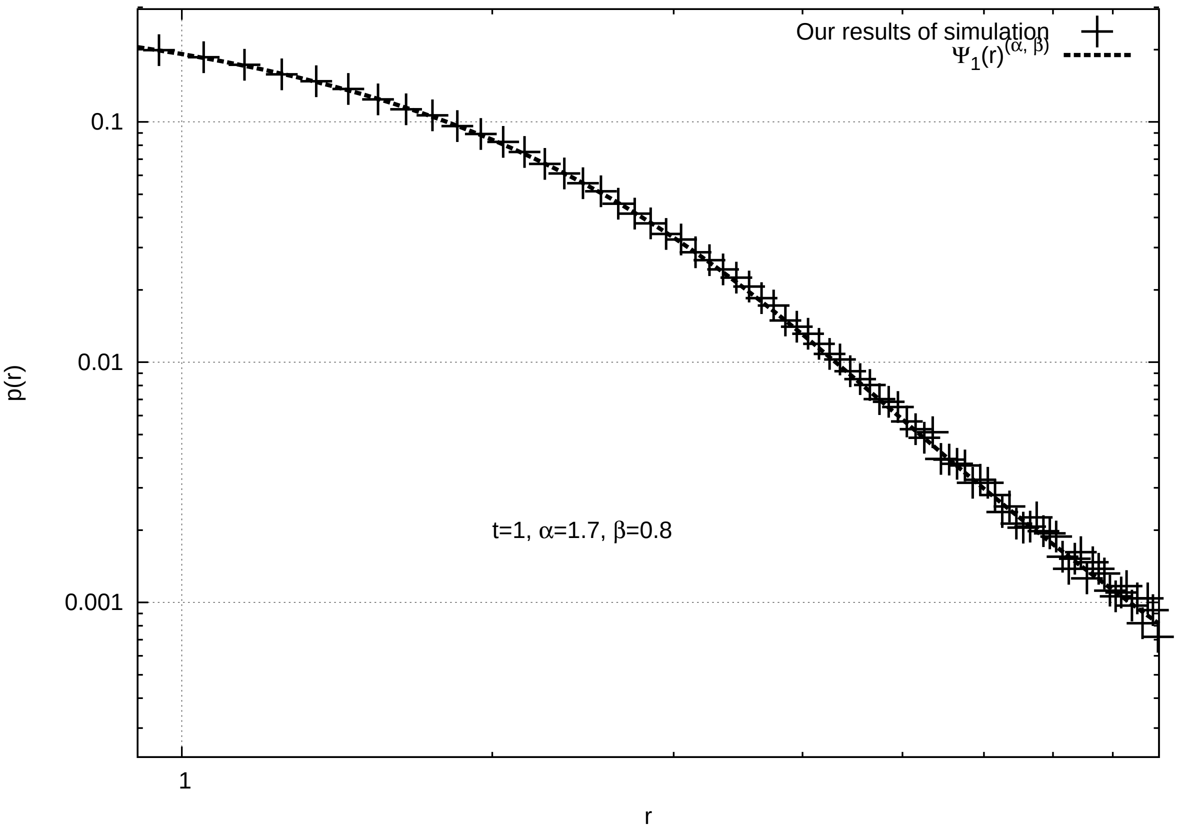

Figure 1. Comparison of the our simulations results with the analytical solution of AD equation.

where the scaling function $\Psi_m^{(\alpha,\beta)}(r)$,

$$\Psi_m^{(\alpha,\beta)}(r) = \int\limits_0^\infty q_m^{(\alpha)}(r\tau^\beta) q_1^{(\beta,1)}(\tau)\tau^{3\beta/\alpha} d\tau,$$

is determined by m-dimensional spherically-symmetrical stable distribution $q_m^{(\alpha)}(r)$ ($\alpha \leq 2$) and one-sided stable distribution $q_1^{(\beta,1)}(t)$ with characteristic exponent β [7, 8].

The continuous-time random walks with a one-parameter Mittag-Leffler distribution of waiting times and a symmetric Lévy α-stable distribution of jumps in space yield the propagator function of the Cauchy problem for a space-time anomalous diffusion equation.

3. Results

We conducted a simulation of random walk processes in one-dimensional case. As seen from (6) simulation result should coincide with the function $\Psi_1^{(\alpha,\beta)}(r)$ from the analytical solutions of the equation (5) for $D = 1$ and $t = 1$. It can be seen good agreement between particle distribution resulting by Monte Carlo simulation with the analytical solution $p(r)$ (Fig.1).

The second figure (Fig.2) shows the spatial distribution of particles in a model of anomalous diffusion at different times, resulting from our Monte-Carlo simulation.

We have also performed simulations in three-dimensional AD walk process with the finite velocity of particles between the inhomogeneities. Estimating the magnitude of the velocity of particles at "knee" of the spectrum of cosmic rays, we concluded from simulation results that the effect of finite velocity on the shape of the spectrum is negligible.

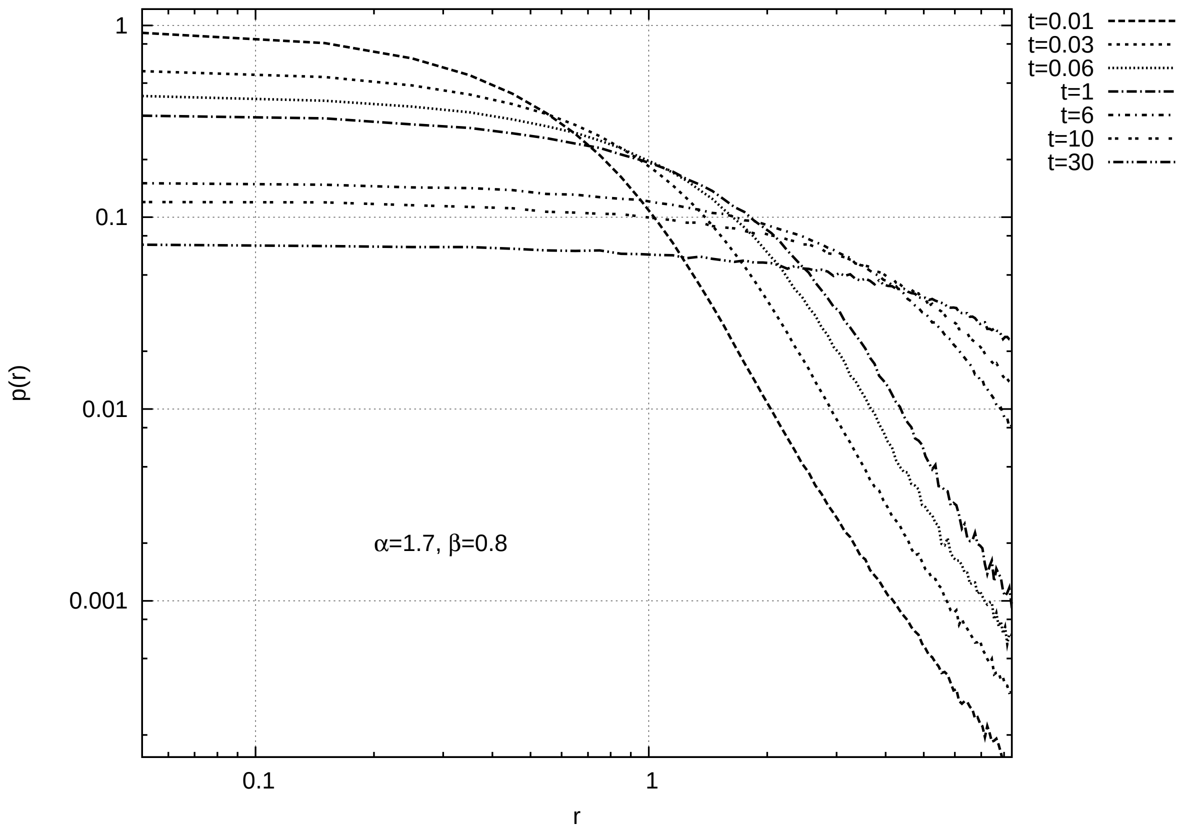

Figure 2. The spatial distribution $p(r)$ of particles at different times t.

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