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Fractional anomalous diffusion models of cosmic ray transport in the Galaxy

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Abstract: A new approach to cosmic ray propagation in the Galaxy based on using differential equation with partial space and time derivatives of fractional orders is discussed, their physical sense is elaborated, some results are analyzed and criticized, direct link to the anisotropy problem is demonstrated and necessary improvements are offered.

Keywords: cosmic ray transport, anomalous diffusion, fractional derivatives, anisotropy

1 Introduction

First fractionally differential model of anomalous cosmic ray (CR) diffusion in Galaxy offered by Lagutin, Nikulin and Uchaikin [1, 2] was based on the differential equation with partial derivatives of fractional order

$$\left[{}_0\mathsf{D}_t^\beta + C(-\triangle)^{\alpha/2}\right]\psi(\mathbf{r},t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta(\mathbf{r}), \quad (1)$$

derived and solved by Saichev and Zaslavsky in one-dimensional case [3] and Zolotarev and Uchaikin in 3-dimensional case [4]. This LNU-model assumed $\alpha = 1.7$ and $\beta = 1 \div 0.8$ could interpret the knee in a high energy part of the energy spectrum as a result of self-similar (fractal) inhomogeneity of interstellar magnetic field.

The LNU-model (with $\alpha = 1$) has been used in [5] for simulation of CR propagation in the interstelar medium (ISM). The authors noted that they prefer this model since (i) the ISM is certainly non-uniform and has quasi-fractal properties, (ii) this model gives the small radial gradient of CR in the Galaxy, (iii) the formation of the Galactic Halo looks more natural in frame of this model etc.

However, Lagutin and Tyumentsev [6] made a crucial step passing to $\alpha = 0.3$ preserving the structure of basic equation and its solution (we refer to this choice of α as LT-model). In frame of LT-model, series of calculations were performed and published in various issues [6, 7].

Recently, Uchaikin has criticized the LT-model and proposed a new more adequate perfect model involving the fractional material derivative which takes into account the boundedness of CR particle velocity [8]. In this report, we continue discussion of U-model started in [9, 10] by using logical, analytical and numerical means.

2 What does mean the fractional equation?

For adequate understanding of Eq.(1), we pass to Fourier-Laplace variables $t \mapsto \lambda,\ \mathbf{r} \mapsto \mathbf{k}$.

$$[b\lambda^\beta + a|\mathbf{k}|^\alpha]\,\tilde{\psi}(\mathbf{k},\lambda) = b\lambda^{\beta-1}. \quad (2)$$

Let us rewrite Eq.(2) in the following asymptotically ($\lambda \to 0$, $|\mathbf{k}| \to 0$) equivalent form:

$$[1-(1-b\lambda^\beta)(1-a|\mathbf{k}|^\alpha)]\,\tilde{\psi}(\mathbf{k},\lambda) = \frac{1-(1-b\lambda^\beta)}{\lambda}. \quad (3)$$

The terms $1-b\lambda^\beta$ and $1-a|\mathbf{k}|^\alpha$ can be considered as main asymptotical terms in integral transforms of some time and space probability densities $q(t)$ and $p(\mathbf{r})$

$$\tilde{q}(\lambda) = \int_0^\infty e^{-\lambda t} q(t)dt \sim 1 - b\lambda^\beta, \quad \lambda \to 0 \quad (4)$$

and

$$\tilde{p}(\mathbf{k}) = \int_{\mathrm{R}^d} e^{-i\mathbf{k}\mathbf{r}} p(\mathbf{r})d\mathbf{r} \sim 1 - a|\mathbf{k}|^\alpha, \quad |\mathbf{k}| \to 0. \quad (5)$$

Thus, Eq. (2) can be considered as the asymptotical representation of some "exact" equation

$$L_\infty(\lambda,\mathbf{k})\tilde{\psi}(\mathbf{k},\lambda) \equiv [1-\tilde{q}(\lambda)\tilde{p}(\mathbf{k})]\tilde{\psi}(\mathbf{k},\lambda) = \tilde{Q}(\lambda), \quad (6)$$

looking in time-space variables as

$$\psi(\mathbf{r},t) = \int_0^t \int d\mathbf{r}'\, q(t')p(\mathbf{r}')\psi(\mathbf{r}-\mathbf{r}',t-t') + \delta(\mathbf{r})Q(t), \quad (2)$$

$$Q(t) = \int_t^\infty q(t')dt'.$$

Eq. (7) describes the following jump process. A particle placed at origin $\mathbf{r} = 0$ at time $t = 0$, stays there during a random time T, which is distributed with the density $q(t)$, and then performs the random jump $0 \mapsto \mathbf{r}' = 0 + \mathbf{R}$ with probability density $p(\mathbf{r}')$. Here, the particle has been waited again for a random time T and then jumps again, and so on. It is very important to stress that 1) waiting time and jump vector are independent of each other, and 2) the jumps are instantaneous, the particle disappears at point $\mathbf{r}$ at time t and appears at point $\mathbf{r} + \mathbf{R}'$ at the same time t independently of the displacement distance $\mathbf{R}'$. In case of normal diffusion ($\alpha = 2$, $\beta = 1$), random variables T and $\mathbf{R}$ become negligibly small in the asymptotical regime, but in case of anomalous diffusion ($\alpha < 2$ or/and $\beta < 1$) conditions (3)–(4) generate long tail distributions of inverse power type, and some of random summands T_j, $\mathbf{R}_j$ are observable at any scales (see, for details, [11]).

3 CR transport from LT model point of view

The model process described by Eq. (1) depends on values of $\alpha \in (0, 2]$ and $\beta \in (0, 1]$. In case $1 < \alpha < 2$ and $\beta \leq 1$ (as it takes in LNU model), we observe moderate jumps at all scales, enhanced spreading the diffusion packet and appearance of long tails in probability distribution. Except these changes, the process is similar to the normal diffusion (Brownian motion). In particular, the ballistic spatial restrictions observed at the beginning of trajectory drop at large time and have an influence on the process only through the diffusion coefficient.

On the contrary, the diffusion packet considered in LT-model grows as t^β/α; this means that the ballistic restrictions do not have influence at short times but essentially restrict the form of the diffusion packet at long times (see, for details, [9]). Schematically, this is shown in Fig. 1.

A more impressive distinction between these models is revealed in real cosmic scales. The ordinary diffusion model and LNU models manifest trajectories looking quite acceptable in this scale while LT trajectories are absolutely unexpected: we see a small cluster of points where the particle wastes its life time and then instantly crosses the Galaxy and disappears

Such long rectilinear instantaneous travels contradict to physical principles and our knowledge about ISM property. It is easy to guess that the anisotropy of CR is the most sensitive to these changes in the process. To verify this assumption we perform Monte Carlo simulation of space-averaged anisotropy parameters of CR propagation from a single point source. The results are: $\bar{\delta}_1 = 3.6 \pm 0.3\%$ in case of ordinary Brownian motion, $\bar{\delta}_1 = 12.42 \pm 0.2\%$ in case of LNU-model and $\bar{\delta}_1 = 97.1 \pm 0.6\%$ in frame of LT-model (with using data given in [6]).

Comments should be done. First, these numbers relate to some model case and no need to be compared with the observation data. Second, nevertheless, they may play an instructive role in understanding the fractional dynamics

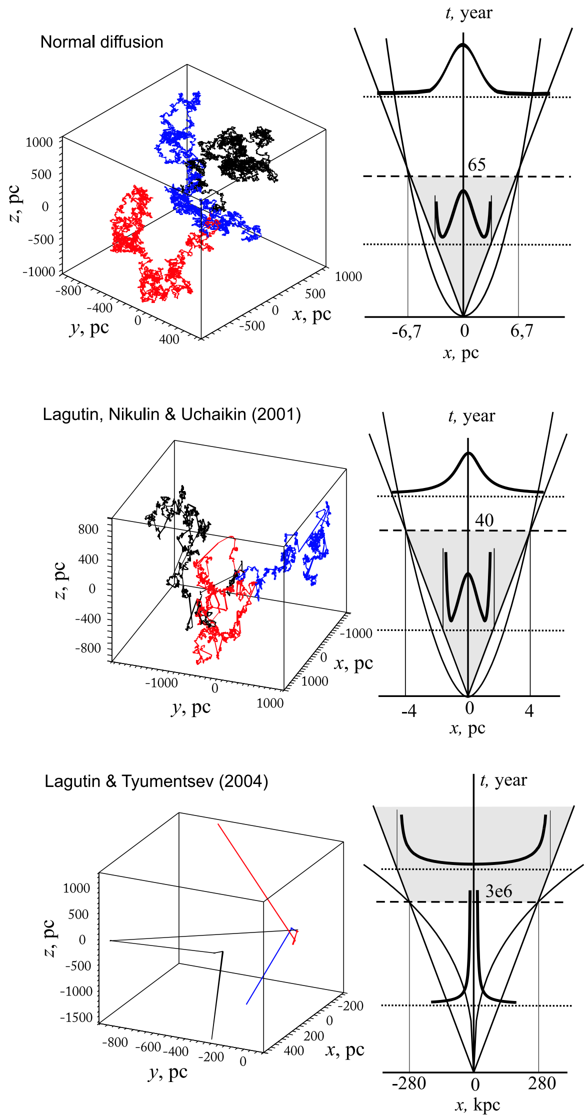

Figure 1: Examples of trajectories in three models: normal diffusion (path length $l = 1$ pc), LNU-model ($\alpha = 1.7, \beta = 0.8$, $D_0 = 2.4 \cdot 10^{-3}$ pc$^{1.7}$/year$^{0.8}$, $D = D_0 R^{0.27}$, $E = 10^6$ GeV), and LT-model ($\alpha = 0.3, \beta = 0.8$, $D_0 = 4 \cdot 10^{-6}$ pc$^{0.3}$/year$^{0.8}$, $D = D_0 R^{0.27}$, $E = 10^6$ GeV). The left panels present schematically the diffusion packet spreading laws, shadowed regions correspond to regions with determining influence of finiteness of velocity.

equation. And third, the low level of anisotropy demonstrated by Lagutin and Tyumentsev [6], namely, $0.05 \div 0.3\%$ for the energy range $10^2 \div 10^9$ GeV, was obtained in a quite simple but not very convincing way. The result was calculated as a ratio of the particle current from a single point source, j_r, to the concentration generated by many sources, $\langle N \rangle = kN$, where the number of sources k was used as a fitting parameter (Eq. (52) of the cited work).

4 Conclusion

As should be clear from the arguments presented above, equations with fractional derivatives possess a high potential for description of CR propagation. Taking into account quasi-fractal structure of ISM presented by index α, and presence of magnetic plugs and traps allows to describe CR propagation in more details and more adequate than the ordinary diffusion. Kermani and Fatemi [12] state that the best fit for the value of α is about 1.65. They compare the results obtained in frames of anomalous diffusion model with cosmic ray spectrum and radial gradient in the vicinity of the solar system. They write that the data on radial gradient allows to use the fit range for α from 1.6 to 1.9.

In our work [9], it has been shown that very important role belongs to the finiteness of the CR particle velocity. We believe that the combination of LNU model with the finite velocity principle [13] leads to creation of a new fruitful model.

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