Combinatorial Quantum Gravity

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In a recently developed approach, geometry is modelled as an emergent property of random networks. Here I show that one of these models I proposed is exactly quantum gravity defined in terms of the combinatorial Ricci curvature recently derived by Ollivier. Geometry in the weak (classical) gravity regime arises in a phase transition driven by the condensation of short graph cycles. The strong (quantum) gravity regime corresponds to "small world" random graphs with logarithmic distance scaling.

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Discrete models of quantum gravity [1] are typically based on (variants of) Regge calculus [2]. Since this measures the curvature of simplicial complexes, space-times are consequently discretized as generalized triangulations. Recently, however a new approach has been developed in which a discrete space-time is not postulated but, rather, it is considered as an emergent property of a random network [3] : the correct space-time is expected to self-organize according to the rules of the network and is thus described by emergent topologies and geometries [4].

First attempts to formulate such an emergent quantum gravity go back to string nets [5] and the models of quantum networks introduced in [6] and recently revisited in [7]. It has also been recently shown that geometry can emerge on the boundary of random tensor networks [8]. In [9, 10] I have proposed a model in which networks with the characteristics of discrete space-time self-assemble form purely combinatorial bits as ground states of an Ising-type model or, in other words, as the most probable configuration of a Gaussian random graph model.

Since these networks are essentially random configurations, the Regge formulation of curvature is no more applicable, a purely combinatorial version of Ricci curvature is needed. Recently, exactly such a combinatorial Ricci curvature has been proposed by Ollivier [11] and further elaborated on in [12]. In this paper I prove that the model proposed in [10] is quantum gravity defined according to this combinatorial Ricci curvature.

The model is formulated in terms of N bits $s_i = \pm 1$ and $N(N-1)/2$ bits $w_{ij} = w_{ji} = 0, 1$, for $i, j = 1 \dots N$. A value $s_i = +1$ denotes the existence of space-time, while $s_i = -1$ indicates the absence of space time (or the presence of anti-space-time). A value $w_{ij} = 1$ denotes a connection between bits s_i and s_j, a value $w_{ij} = 0$ indicates that s_i and s_j are not connected. These link variables are symmetric, $w_{ij} = w_{ji}$ and vanish on the diagonal, $w_{ii} = 0$.

The Hamiltonian for the coupled Ising-network system is (I use natural units $c = 1$, $\hbar = 1$)

$$
\begin{aligned}
H &= H_0 + H_E \ , \\
H_0 &= \frac{J}{2} \sum_{i \neq j} \sum_{\substack{k \neq i \\ k \neq j}} w_{ik} w_{kj} - \frac{1}{2} \sum_{i \neq j} s_i w_{ij} s_j \ , \\
H_E &= -\frac{1}{2g} \left[\mathrm{Tr}\left(w^4\right) - 2 \sum_i \left(w_{ii}^2\right)^2 + \sum_i w_{ii}^2 \right] \ , \quad (1)
\end{aligned}
$$

where the Ising coupling, representing the Planck energy scale is also set to 1 for simplicity. J and g are then the two dimensionless coupling constants of the model.

The second term in the Hamiltonian H_0 is the standard ferromagnetic Ising model. The first term, instead represents a nearest-neighbours antiferromagnetic Ising model for the links, with "nearest-neighbours" meant in the sense that only links sharing a common vertex are coupled. In absence of the antiferromagnetic link term, and with the links w_{ij} uniformly drawn from random adjacency matrices of degree 4 , the model would be Kazakov's random lattice Ising model in two dimensions [14]. The generalizations with respect to Kazakov's model, thus consist in dropping the restriction to degree 4 and drawing the random adjacency matrices from a Gaussian distribution. The model H_0 is also closely related to the painted graphs of [15].

The competition between the vertex ferromagnetic coupling and the link antiferromagnetic one generates frustration in the model. First of all, vertices at the ends of a non-vanishing link tend to align because of the standard ferromagnetic coupling. This same coupling favours then the creation of many links in a vertex-aligned state, corresponding to a fully formed space-time. On the other side, due to the antiferromagnetic link coupling, creating many links costs energy. The compromise is to create a uniform, optimal number of links per vertex depending on the coupling constant J, generating thus a connected random regular graph with adjacency matrix w and connectivity $2d$ for $J = 1/4d - 1$ as I have derived in [9, 10]. The first coupling constant J determines thus essentially the dimensionality of the emerging space-time.

The elementary excitations above this ground state are obtained by adding or subtracting one link and are given by $\Delta_{\rm add} E_{ij} = 2d/(4d-1) - 1/2$ and $\Delta_{\rm elim} E_{ij} =$

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$1/2-(2d-1)/(4d-1)$ [9, 10]. For large d these excitation energies behave as $O(1/8d)$ but they are $O(1)$ for the smallest d like $d=2,3$. For sufficiently large values of g one can then consider H_E as a perturbation which is just lifting the large degeneracy in the ground state manifold of H_0, i.e. random regular graphs (note that, in this space, the second and third terms in H_E reduce simply to the constants $-8Nd^2$ and $2dN$ since $w^2_{ii}=2d$, for all vertices i). The formation of a random regular graph from a disordered soup of bits at high (stochastic) temperature can be considered as the transition from combinatorics to topology, in the sense that well-defined neighbourhood relations are established. As I now show, the gravitational coupling g governs the transition from topology to geometry.

Random regular graphs [16] are "small worlds", i.e. their diameter and average distances on the graphs scale logarithmically with the number N of vertices (the volume). This behaviour is clearly unsuitable to model a geometric space-time. Random regular graphs have locally a tree structure with very sparse short cycles governed by a Poisson distribution [16] with mean $(2d-1)^l/2l$ for cycles of length l. Loosely speaking, the lack of links used to form short cycles leaves lots of links available to form "shortcuts" among otherwise "distant" parts of the graph, causing the logarithmic scaling behaviour. From a graph theory point of view, the energy term H_E in (1), favouring the formation of squares (4-cycles), "uses up" lots of links to make short cycles so that the graph "opens up" to become a "large world" with graph distances scaling as inverse integer powers of volume. I will now show what the significance of this term is from a physics point of view.

To do so I will introduce the concept of combinatorial Ricci curvature on generic graphs [11–13]. As in the continuum Ricci curvature is associated with a point and a direction on a manifold, its discrete version is associated with a vertex i and a link e_i of a graph. Averaging over all links emanating from a vertex gives the discrete version of the Ricci scalar at that vertex. From a geodesic transport point of view, the Ricci curvature can be thought of as a measure of how much (infinitesimal) spheres (or balls) around a point contract (positive Ricci curvature) or expand (negative Ricci curvature) when they are transported along a geodesic with a given tangent vector at the point under consideration. The Ollivier curvature is a discrete version of the same measure. For two vertices i and $j=i+e_i$ it compares the Wasserstein (or earth-mover) distance $W(\mu_i,\mu_j)$ between the two uniform probability measures $\mu_{i,j}$ on the spheres around i and j to the distance $d(i,j)$ on the graph and is defined as

$$\kappa(i,j)=1-\frac{W(\mu_i,\mu_j)}{d(i,j)}\ . \tag{2}$$

The Wasserstein distance between two probability measures μ_1 and μ_2 on the graph is defined as

$$W(\mu_1,\mu_2)=\inf\sum_{i,j}\xi(i,j)d(i,j)\ , \tag{3}$$

where the infimum has to be taken over all couplings (or transference plans) $\xi(i,j)$ i.e. over all plans on how to transport a unit mass distributed according to μ_1 around i to the same mass distributed according to μ_2 around j,

$$\sum_j\xi(i,j)=\mu_1(i)\ ,\qquad \sum_i\xi(i,j)=\mu_2(j)\ . \tag{4}$$

The Ollivier curvature is very intuitive but, in general not easy to compute and work with. To help this, I will introduce a small modification to H_0 which is inessential but makes things simpler and analytically tractable. This consists in adding a term

$$H_0\to H_0+\sum_{k=1}^{\infty}\mathrm{Tr}\left(w^{2k+1}\right)\ , \tag{5}$$

which suppresses all odd cycles in the ground state manifold. Graphs with no odd cycles are bipartite and for bipartite regular graphs the Ollivier Ricci curvature simplifies considerably [13]. As in the case of traditional antiferromagnetic spin systems, the restriction to bipartite structures renders the system amenable to analytical analysis. In the present case, however, the restriction is quite harmless since, as it is easy to convince oneself, the Ollivier Ricci curvature scalar at one vertex is influenced only by triangles, squares and pentagrams passing through the vertex (which is a graph version of locality embodied by dependence on first and second derivatives only) and all these short loops are anyhow essentially absent in random regular graphs (the probability of such a short loop passing through a vertex vanishes for $N\to\infty$).

The Ollivier Ricci curvature of an edge (ij) of a regular bipartite graph with connectivity $2d$ is given by [13],

$$\begin{aligned}\kappa(i,j)\quad &=-\frac{1}{d}\Big[(2d-2)-|N_1(j)|\\ &+\sum_a\left(|L_a(j)|-|U_a(i)|\right)\times\mathbf{1}_{\{|U_a(i)|<|L_a(j)|\}}\Big]_+\end{aligned} \tag{6}$$

where $N_1(i)$ denotes the set of neighbours of i which are on a 4-cycle supported on (ij), $\mathbf{1}$ denotes the indicator function (1 if the corresponding condition is satisfied, 0 otherwise) and the undescript "+" denotes $z_+=Max(z,0)$ so that the Ollivier Ricci curvature for bipartite graphs is always zero or negative. Suppose that $R(i,j)$ is the subgraph induced by $N_1(i)\cup N_1(j)$ and $R_1(i,j)...R_q(i,j)$are the connected components of $R(i,j)$. Then $U_a(i)=R_a(i,j)\cap N_1(i)$ and $L_a(j)=R_a(i,j)\cap N_1(j)$ for $a=1\ldots q$.

This expression still looks forbidding but is in reality quite simple. Two different squares (4-cycles) on a connected regular graph can either share 0 edges, if they are separated, or 1 edge or 2 edges if they touch. It is easy to convince oneself that the second term, involving the sum of connected component of a subgraph only contributes for squares that share 2 edges. Indeed, for an isolated square $|N_1| = 1$ for all vertices on the square. If an edge supports N_s squares which do not share another edge, then $|N_1(i)| = |N_1(j)| = N_s$ and $|U_a(i)| = |L_a(j)|$ since all the vertices within $N_1(i)$ and $N_1(j)$ are disconnected because of the absence of triangles in a bipartite graphs and all the vertices of $N_1(i)$ are disconnected from those in $N_1(j)$ since, by assumption, the edge does not support two different squares sharing two edges.

In a random regular bipartite graph squares are extremely sparse, being distributed according to a Poisson distribution with fixed mean $(2d-1)^4/4$ [17] so that the probability of finding one vanishes for $N \to \infty$. If the coupling constant g is sufficiently large, the H_E term in the Hamiltonian will induce additional squares but typically not enough to reach with high probability dense configurations in which there are lots of squares touching on two edges. In this regime one can thus safely ignore squares sharing two edges, in which case the Ollivier Ricci curvature for $2d$-regular bipartite squares reduces to

$$\kappa(i,j) = -\frac{1}{d}\big[(2d-2) - N_s(ij)\big] \ , \tag{7}$$

where $N_s(ij)$ is the total number of squares supported on edge (ij). Note also that I have left out the subscript "+". This is because, for squares sharing maximally one edge $N_s(ij) \le (2d-2)$, as I now show.

To do so, let me consider the uniform configuration with maximum square density. First observe that, by the degree sum formula $2e = \sum_{i\ge 3} i\ v_i$, with e the number of edges and v_i the number of vertices of degree i, one can derive that $2d$-regular graphs have exactly dN edges. This means that one can uniquely assign to each vertex exactly d edges. Out of d edges one can form at most $d(d-1)/2$ different squares that share maximally one edge. Therefore the total number of squares is $Nd(d-1)/2$ squares, each vertex having $d(d-1)/2$ squares uniquely assigned to it. Since a square is made of four vertices and four edges and there are a total of N vertices and dN edges, this means that each vertex is shared by exactly $2d(d-1)$ squares and each edge is shared by $2d-2$ squares. Thus, in this uniform configuration with maximum number of squares (sharing at most one edge) each edge supports exactly $2d-2$ squares, which shows that indeed $N_s(ij) \le 2d-2$. The maximum value $N_s(ij) = 2d-2$ for all edges is realized in Ricci flat, locally Euclidean graphs with neighbourhoods locally homeomorphic to $\mathbb{Z}^d$.

The "integral" of the Ollivier Ricci curvature scalar over the graph is

$$\begin{aligned}\sum_i \kappa(i) &= -\frac{2d-2}{d}N + \frac{1}{d^2}\sum_i \sum_{e_i} N(e_i) \\ &= \frac{-4}{d^2}\left[\frac{d(d-1)}{2}N - N_s\right] \ ,\end{aligned} \tag{8}$$

with N_s the total number of squares on the graph. The factor 4 comes from the fact that each square is shared by four vertices. On the other side, the total number of squares on a graph is given by [18]

$$N_s = \frac{1}{8}\left[\mathrm{Tr}\left(w^4\right) - 2e - 2\sum_i k_i(k_i-1)\right] \ , \tag{9}$$

where e is the total number of edges and k_i are the vertex connectivities. For the $2d$-regular graphs of interest here this reduces to

$$\begin{aligned} N_s &= \frac{1}{8}\left[\mathrm{Tr}\left(w^4\right) - 8Nd^2 + 2dN\right] \\ &= \frac{1}{8}\left[\mathrm{Tr}\left(w^4\right) - 2\sum_i \left(w_{ii}^2\right)^2 + \sum_i w_{ii}^2\right] \ ,\end{aligned} \tag{10}$$

where I have used that $2d = w_{ii}^2$ is the uniform vertex degree. Finally, one can combine (8), (10) and (1) to obtain

$$H_E = -\frac{d^2}{g}\Big[\sum_i \kappa(i) + \frac{2d-2}{d}N\Big] \ , \tag{11}$$

which shows that the term H_E, favouring the formation of squares on the graph is nothing else than a combinatorial version of the Einstein action (apart from an irrelevant constant). Indeed, sampling random regular bipartite graphs (rrbg) according to the Boltzmann probability

$$\begin{aligned} p_B &= \frac{\exp\left(-H_E\right)}{\sum_{\mathrm{rrbg}} \exp\left(-H_E\right)} = \frac{\exp\left(\frac{d^2}{g}\sum_i \kappa(i)\right)}{Z} \ , \\ Z &= \sum_{\mathrm{rrbg}} \exp\left(\frac{d^2}{g}\sum_i \kappa(i)\right) \ ,\end{aligned} \tag{12}$$

amounts exactly to computing the combinatorial quantum gravity partition function. Note that the constant term in (11) drops out from this expression and that the combinatorial Ollivier Ricci curvature scalar is always negative, making the sum well defined also for $N \to \infty$.

The asymptotic number $|\mathcal{G}^b_{N,2d}|$ of random $2d$-regular bipartite graphs on N vertices is known [17],

$$|\mathcal{G}^b_{N,2d}| = \frac{(dN)!\ e^{-\frac{1}{2}(2d-1)^2}}{\left((2d)!\right)^N} \propto e^{dN\ln N} \ , \tag{13}$$

for $N \gg d$. There is, instead essentially only one fully ordered "$\mathbb{Z}^d$ configuration" with $N_s = N$. For large N, the free energy is thus given by

$$F = \frac{4}{g}\big[\frac{d(d-1)}{2}N - N_s\big] - S\left(N_s\right) \ , \tag{14}$$

where, with the approximation of ignoring squares sharing two edges, $0 \le N_s \le (d(d-1)/2)N$ and the entropy satisfies

$$\lim_{N_s \to d(d-1)N/2} S\left(N_s\right) = 0 \ , \quad \lim_{N_s \to (2d-1)^4/4} S\left(N_s\right) = dN\ln N \ . \tag{15}$$

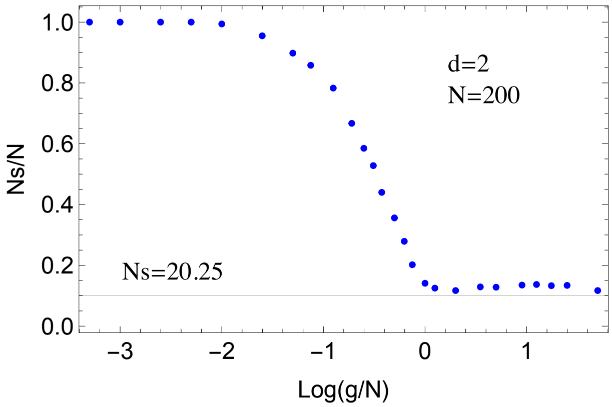

FIG. 1: Monte Carlo simulation of the average number of squares for $d = 2$ and $N = 200$. Random regular graphs with sparse squares $N_s \sim$ Poisson (20.25) and logarithmic distance scaling at large values of the gravitational coupling constant turn into $\mathbb{Z}^2$ graphs with the maximum number of squares $N_s = N$ and power-law distances when gravitation becomes weak.

In the strong (quantum) gravity regime $g \gg N$, the energy term in the free energy is always overwhelmed by the entropy and the typical configuration is that of a random regular bipartite graph. In this regime squares (and all other short cycles) are sparse, distributed according to a Poisson distribution with mean $(2d-1)^4/4$, e.g. 20.25 for $d = 2$ [17] and graph distances scale logarithmically with the volume N. When gravity becomes weaker (classical), $g \ll (2d-2)/\ln N$, the energy term dominates the free energy and the typical configuration is one with the minimum energy, i.e. with the maximum number of squares $N_s = (d(d-1)/2)N$. This is a Ricci flat, locally Euclidean configuration with neighbourhoods homeomorphic to $\mathbb{Z}^d$ and graph distances scaling as $N^{1/d}$. In between these two extremal regimes one can expect a phase transition in which squares condense and geometry emerges from a purely random configuration. This transition, with the average number of squares (4-cycles) as the order parameter is shown in Fig. 1 for $d = 2$ and $N = 200$ in a Metropolis Monte-Carlo simulation. This suggests a second-order transition with critical point $g_c = O(N)$, which would define the model non-perturbatively. As anticipated, within the topology phase (with well defined neighbourhood relations only) the emergence of geometry from random links is governed by the gravity coupling constant g and induced by the condensation of the smallest possible cycles. A similar conclusion has been reached very recently in [19] where it was shown that, in graphs where triangles are admitted as shortest cycles, geometry is tied to the clustering coefficient.

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[1] For a review see: J. Ambjorn, A. Görlich, J. Jurkiewicz and R. Loll, *Phy. Rep.* **58** (2012) 127.
[2] T. Regge, *Nuovo Cim.* **19** (1961) 558; for a review see e.g.: R. M. Williams and P. A. Tuckey, *Class. Quant. Gravity* **9** (1992) 1409.
[3] For a comprehensive review see: R. Albert and L. Barabasi, *Rev. Mod. Phys.* **74** (2002) 47.
[4] For a review see: G. Bianconi and C. Rahmede, *Phys. Rev.* **E93** (2016) 032315.
[5] M. Levin and W-G. Wen, *Phys. Rev.* **71** (2004) 045110.
[6] T. Konopka, F. Markopoulou and L. Smolin, arXiv:hep-th / 0611197; T. Konopka, F. Markopoulou and S. Severini, *Phys. Rev.* **D77** (2008) 104029.
[7] G. Bianconi, C. Rahmede and Z. Wu, *Phys. Rev.* **E92** (2105) 022815; G. Bianconi and C. Rahmede, *Scient. Rep* **5** (2015) 13979.
[8] H. Chen, N. Sasakura and Y. Sato, *Phys. Rev.* **D93** (2016) 064071.
[9] C. A. Trugenberger, *Phys. Rev.* **D92** (2015) 084014, *Phys. Rev.* **E92** (2015) 062818.
[10] C. A. Trugenberger, arXiv:1610.05339, to appear in *Phys. Rev.* **E.**
[11] Y. Ollivier, *J. Funct. Anal.* **256** (2009) 810; Y. Ollivier, *Adv. Stud. Pure Math.* **57** (2010) 343;
[12] Y. Linn, L. Lu and S. T. Yau, *Tohoku Math. J.* **63** (2011) 605; B. Loisel and P. Romon, *Axioms* **3** (2014) 119; J. Jost and S. Liu, *Discrete Comput. Geom.* **51** (2014) 300.
[13] B. B. Bhattacharya and S. Mukherjee, *Discrete Mathematics* **338** (2015) 23.
[14] V. A. Kazakov, *Phys. Lett.* **A119** (1986) 140; D. V. Boulatov and V. A. Kazakov, *Phys. Lett.* **B186** (1987) 379.
[15] V. Avetisov, A. Gorsky, S. Nechaev and O. Valba, *Phys. Rev.* **E93** (2015) 012302.
[16] B. Bollobas and W. Fernandez de la Vega, *Combinatorica* **2** (1981) 125; N. C. Wormald, *Surveys in Combinatorics*, London Mathematical Society Lecture Note Series, J. D. Lamb and D. A. Preece eds. Cambridge University Press, Cambridge (1999); S. N. Dorogotsev, *Lectures on Complex Networks*, Clarendon Press, Oxford (2010).
[17] P. E. O'Neil, *Bull. Am. Math. Soc.* **75** (1969) 1276; N.C. Wormald, *Surveys in Combinatorics*, LMS Lectures Note Series **267**, J. D. Lamb and D. A. Pierce eds., (1999) 239.
[18] F. Harary and B. Manvel, *Mat. Casopis Slov. Akad. Vied* **21** (1971) 55.
[19] D. Krioukov, *Phys. Rev. Lett.* **116** (2016) 208302.