Selection Postulates and Probability Rules in the Problem of Quantum Measurement

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Abstract

Various approaches to quantum measurement problem within the framework of usual unitary quantum dynamics are considered. It is argued that neither decoherence theory nor many-worlds interpretation of quantum mechanics do provide ultimate solution of the measurement problem: they cannot solve the problem of an alternative selection within the framework of unitary quantum dynamics. It is argued that the selection postulate in quantum theory is a very fundamental entity tightly connected with the nature of mathematics and with the nature of the mind, while the probability rules are more technical things admitting various approaches based on various sets of axioms.

Key Words: quantum measurement problem, projection postulate, decoherence theory, mind, consciousness

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1. Introduction

The quantum measurement problem

The essence of the quantum measurement problem is in the following. The main quantum dynamical equation—Shrödinder equation—and evolution governed by this equation are essentially linear. Therefore, having initial quantum system under measurement S in a state of superposition of some orthogonal eigenstates (say $|S_a\rangle$ and $|S_b\rangle$) and supposing the measurement process to be governed by the Shrödinder equation, we inevitably obtain that the final measurement state must be a linear superposition of various measurement results (a and b in this particular situation). But actually only one of all possible results of the measurement is accepted by an observer—a *or* b, not both a and b at the same time. The problem is that the dynamical equations of quantum mechanics do not provide a mechanism of selection a single result of measurement.

Mathematically the problem (in the above mentioned simple situation) may be written as following. Let the state of the system S before measurement be $\alpha|S_a\rangle+\beta|S_b\rangle$ and the state of the device D before measurement be $|D_0\rangle$. The device can produce the results a or b represented by the final states of the device $|D_a\rangle$ and $|D_b\rangle$ respectively. Solution of Shrödinder equation provides then a *linear* (moreover—unitary) operator U that describes evolution of the composite system $S\otimes D$ during the measurement. The device is a “good” device to measure a or b if

$$U|S_a\rangle|D_0\rangle=|S_a\rangle|D_a\rangle;U|S_b\rangle|D_0\rangle=|S_b\rangle|D_b\rangle. \quad \textbf{(1)}$$

The device D of this kind provides the best possible measurement of the observable represented by the values $\{a,b\}$ in the sense of the minimal disturbance of the measured system S. The related measurement is called *ideal measurement* or *measurement of the first type*.

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It is sufficient to consider only ideal measurements to trace the problem of quantum measurement. It follows from (1) and linearity of operator U that

$$U(\alpha|S_a\rangle+\beta|S_b\rangle)|D_0\rangle=\alpha|S_a\rangle|D_a\rangle+\beta|S_b\rangle|D_b\rangle. \quad \textbf{(2)}$$

It is clearly seen from (2) that if the system S was in a superposition of the eigenstates before the measurement then both outcomes a and b (both pointer positions) are presented in the final state of the measurement. What is the mechanism that selects a single alternative for an observer?

One possible solution of this problem is known long ago. It is so-called Copenhagen interpretation of quantum mechanics that introduces a notion of *projection postulate* or a postulate of *collapse of a quantum state*. This interpretation lays in the basis of most textbooks on quantum mechanics. Projection postulate proclaims that in a measurement like (2) the system $S\otimes D$ after the measurement actually *will be* either in the state $|S_a\rangle|D_a\rangle$ (with probability $|\alpha|^2$, and the result of measurement will be a) or in the state $|S_b\rangle|D_b\rangle$ (with probability $|\beta|^2$, the result being b) and no linear superposition of the eigenstates exist after the measurement have finished.

In the mathematical language of quantum mechanics projection postulate means that the initial ensemble representing the initial *pure* quatum state evolves during the measurement into an ensemble described by a so-called mixed state. This evolution is neither linear nor reversible, contrary to the evolution predicted by Shrödinger equation. There are no direct logical contradictions in such approach but it looks unsatisfactory because it sharply separates measurements (that are not subject of linear Shrödinger evolution) from all other “regular” physical processes (those governed by linear Shrödinger equation) without exact clarification what is a measurement process and what is not. This unsatisfactory situation may be considered as another formulation of the quantum measurement problem. Therefore, the measurement problem may be formulated first, as the problem of selection of alternative and second, if we appeal to projection postulate, as coexistence of two different evolution laws - regular, governed by Shrödinger equation and a special law to describe measurements - projection postulate.

Note that the projection postulate itself proclaims two sharply different things. First, it proclaims that the result of a measurement is some single pure state from a complete set of orthogonal states of the measured system - it is *selection part* of the projection postulate. Second, it predicts the probabilities to find these outcomes - this is *probabilistic part* of the projection postulate (Born's rule for probabilities). We will see that these two aspects are characteristic of all approaches in interpretation of quantum mechanics and have sharply different status in quantum theory.

Other ways to solve the problem of quantum measurement within traditional linear quantum mechanics (besides Copenhagen interpretation) are proposed by so-called decoherence theory (Guilini et al, 1996) and many-worlds interpretation (Everett, 1957; DeWitt and Graham, 1973). There exist also ways connected with no-linear modifications of the Shrödinger equation but they are beyond standard quantum mechanics and we do not touch them in the present paper.

2. Decoherence theory and the quantum measurement problem

Solving quantum measurement problem in decoherence theory may be presented in the following way. Consider first *classical* probabilistic experiment---situation *without* the problem of selection of an alternative. Let possible outcomes of the experiment be a and b and they take place with the probabilities p_a and p_b respectively. It is known how this situation may be described in quantum mechanics. The following description is somewhat schematic, but no essential details are missed. Let outcomes a and b of the experiment correspond to the final states $|C_a\rangle$ and $|C_b\rangle$ respectively of some classical system C (for example, a coin that was tossed up). Then the statement “a has probability p_a and b has probability p_b” means exactly “$|C_a\rangle$ has probability p_a and $|C_b\rangle$ has probability p_b”. But the last statement quantum-mechanically means that the system C is in the mixed state described by the diagonal density operator

$$\rho_C = |C_a\rangle p_a \langle C_a| + |C_b\rangle p_b \langle C_b|. \quad (3)$$

More precisely, we may consider that just after the test has been finished but before the observer's reading off the result, one must describe the system C by the state ρ_C. Alternatively, we may consider that the state ρ_C describes the complete ensemble of the results of all tests of the probabilistic experiment. Thus, it is equally correct to say that a classical system C is in the state like (3) and to say that the outcome a has classical probability p_a and the outcome b has classical probability p_b. Both these statements mean no problem of selecting an alternative because the experiment is completely classical. It is very important that there is no other way to represent classical probability (meaning no selection problem) in quantum mechanics. The representation (3) is the maximum that we can obtain in quantum mechanics for this purpose.

Now return to the quantum measurement described above. It is very important that the observer has no direct perception of the quantum system S and its states. It is worthwile to say that the quantum system lays beyond the horizon of perception of the observer. The only available thing for the observer is the classical device and its states. Formally, an observer must describe quantum measurement as a particular process with some classical setups needed first "to prepare a quantum system" and then "to measure it" (the words "to prepare" and "to measure" are used only for convenience).

Thus the observer's mind or her or his perception during quantum measurement deals exclusively with the states of the device D. The device D itself has no pure quantum state in the final entangled state of the measurement like (2). Instead, the state of the device is described by the density operator obtained as a trace (averaging) on all states of the system S for the final state of the measurement. Such an operator is called the reduced density operator or reduced density matrix. If the complete final state of the measurement (2) is also written in the form of a density operator

$$\begin{aligned}\rho_{SD} &= (\alpha|S_a\rangle|D_a\rangle + \beta|S_b\rangle|D_b\rangle)\times \\ &(\alpha^*\langle S_a|\langle D_a| + \beta^*\langle S_b|\langle D_b|)\end{aligned} \quad (4)$$

then the state of the device itself is

$$\begin{aligned}\rho_D &= \langle S_a|\rho_{SD}|S_a\rangle + \langle S_b|\rho_{SD}|S_b\rangle \\ &= |D_a\rangle|\alpha|^2\langle D_a| + |D_b\rangle|\beta|^2\langle D_b|.\end{aligned} \quad (5)$$

The device D is a classical object and the form of its state (5) occurs to be the same as for the classical system C in the classical probability experiment, (3). But we have already established above that the form of state (3) (and (5) as well) is equivalent to the situation of classical probability test without any problem of selection of alternative. Consequently, during quantum measurement an observer, from its own point of view, does not meet a problem of selection of an alternative - just as in a classical probability experiment. That is, the mixed state of the device in the final state of the measurement solves the problem of selection of an alternative since it reduces this problem to the situation of a classical probabilistic test. Also there is no evident collapse or reduction of state - this may be a solution of complete measurement problem. The core of this argumentation is that it is unimportant what is the origin of a mixed state of some classical system: classical probabilistic experiment like (3) or separation of a mixed state from some pure quantum entangled state like in (5). From the observer's point of view these situations are equivalent. A diagonal density operator, irrespectively of its origin, is equivalent to some classical probability mixture.

The above consideration was simplified in a number of points. For example, we completely neglected interaction of the device D with its environment. Sometimes just decoherence of the state of the device arising due to interaction of the device with the environment is the main subject of the analysis in the decoherence theory (see, for example, the chapter written by E. Joos in the book (Guilini et al, 1996). However, interaction of the device with the measured microsystem is already a sufficient cause of a classical final state of the device in a measurement. Also, in some cases human sensory organs (for example, an eye) may play a role of the device. It may be shown that different technical improvements of the consideration do not change the conclusion. The key features that lead to the solution of the problem are, first, entanglement of states of the measured system S and the device D during measurement like in (2), second, existence of the perception horizon of the human consciousness restricted only by the classical objects, and third, uniqueness of the representation of classical probability in quantum mechanical formalisms by a diagonal density operator.

3. Projection postulate and decoherence theory

The decoherence theory provides more deep view on the quantum measurement problem than the Copenhagen interpretation does. Indeed, the decoherence theory provides a dynamical description of quantum measurement and tries to derive selection of an alternative from unitary dynamics of the composite system, not from an explicit extradynamical postulate. But does the consideration above actually provide an ultimate solution of the quantum measurement problem?

Subtle points of the consideration of section 2 are formula (3) representing a state of classical statistical mixture and formula (5) for the reduced density operator of the device D in the final entangled state of the measurement. The problem is that the notion of density operator actually is connected with the projection postulate, therefore formulas like (3) and (5) make use of the projection postulate in an implicit way. This fact is known and was pointed out, for example, in (Guilini et al, 1996), p. 16 and p. 37, and (Zurek, 2007b). Let us discuss this point explicitly.

3.1 Density operator and mean value law

Both formulas (3) and (5) are implications of the general quantum mechanical formula for the mean value of an observable M in any quantum state $|\Psi\rangle$:

$$\langle M\rangle = \langle\Psi|M|\Psi\rangle. \quad (6)$$

Moreover, the mean value law (6) is a source of the notion of density operator. Proof of these statements is straightforward.

Suppose first that we have an ensemble representing a statistical mixture of some system C in orthogonal states $|C_i\rangle$ with probabilities p_i. Let M be any observable on the system C. Then for the mean value of M we obtain from (6)

$$\langle M\rangle = \sum_i p_i \langle C_i|M|C_i\rangle = Tr(\rho_C M), \quad (7)$$

where

$$\rho_C = \sum_i |C_i\rangle p_i \langle C_i|. \quad (8)$$

Equation (7) shows that the state of the system C in a statistical mixture is represented by the diagonal density operator ρ_C (because the mean value of any observable M is calculated through ρ_C) and (8) coincides with the formula (3) that has been used above to represent a classical probabilistic experiment. Note also that the mean value law (6) is a special case of the formula (7) (with only one of p_i equal to unity and all the rest zero) and therefore may be derived from (7) . Thus (6) and (7) are equivalent.

Consider now the following state of the system $S\otimes D$

$$|\Psi\rangle = \sum_i \alpha_i |S_i\rangle |D_i\rangle \quad (9)$$

and some observable M of the system D. Since this observable belongs to the system D only, it may be written in the form:

$$M = I_S \otimes M_D, \quad (10)$$

where I_S is the unit operator in the space of states of system S, and M_D is an operator in the space of states of system D. Now find the mean value of M in state $|\Psi\rangle$. Using general law (6) , with (9) and (10) , we have

$$\langle M\rangle = \sum_i |\alpha_i|^2 \langle D_i|M_D|D_i\rangle = Tr(\rho_D M_D), \quad (11)$$

where

$$\rho_D = \sum_i \langle S_i|\Psi\rangle\langle\Psi|S_i\rangle \equiv Tr_S(|\Psi\rangle\langle\Psi|). \quad (12)$$

Arguing just as above, we see that ρ_D is the state of the system D and (12) is an exact expression of ρ_D through the pure state $|\Psi\rangle$ of the complete (composite) system. The form of the density operator (12) coincides with the formula (5) that was used in decoherence theory to solve the measurement problem. Thus both formulas (3) and (5) are implication of the mean value law (6) .

3.2 Mean value law, projection postulate and measurements

Now let us prove that the mean value law (6) is equivalent to the reduction postulate. To prove this statement, let us first accurately define notions of *ideal measurement* and the relevant notion of *observable*. Some fine points connected with the problem of selection of an alternative actually have roots in the notion of observable or measurement.

The main feature of the notion of measurement is that a measurement produces just one definite result each time when it is applied: one definite real number. However, quantum linear evolution generally does not lead to outcomes of this type. Therefore, if we want anyway to use notions of measurement or observable, we must postulate some kind of selection of an alternative from the very beginning. Notions of measurement and observable would be meaningless without such postulate. The following definitions represent one possible way to fix and justify these selection rules.

Let S be a quantum system with (for simplicity finite-dimensional) Hilbert space of states $\mathcal{H}$. Then:

Definition 3.2.1 *Each ideal measurement* $\mathcal{M}$ *is unambiguously characterized by an orthonormal basis* $\{|S_j\rangle\}$ *of* $\mathcal{H}$. *For any initial state* $|\Psi\rangle$ *of the system* S *the result of the measurement is one of the states from the set of states* $\{|S_j\rangle\}$. *If the result of the measurement is* $|S_i\rangle$ *then the system* S *is in the state* $|S_i\rangle$ *just after the measurement.*

Definition 3.2.2 *An observable* M *is connected with the measurement* $\mathcal{M}$ *if some real numbers* m_i *correspond to the states* $|S_i\rangle$. *By definition,* M *takes value* m_i *if the result of the measurement* $\mathcal{M}$ *is* $|S_i\rangle$.

Obviously, the observable is unambiguously defined by the operator

$$M = \sum_i m_i |S_i\rangle\langle S_i|. \quad \textbf{(13)}$$

so m_i and $|S_i\rangle$ are eigenvalues and eigenvectors of M respectively. The above definitions explicitly proclaim existence of a selection of an alternative during measurement. The observable plays the role of a scale in respect to the measurement.

Now consider a quantum system S in the state

$$|\Psi\rangle = \sum_i \alpha_i |S_i\rangle, \quad \textbf{(14)}$$

where $|S_i\rangle$ form an orthonormal basis of the system. The reduction postulate claims that during the measurement we obtain the state $|S_i\rangle$ with the probability $|\alpha_i|^2$.

First prove that the mean value law (6) implies the projection postulate. The selection part of projection postulate have been included already in the definition of measurement 3.2.1 so we should prove the probability part. Let us consider the observable described by the operator $P_j = |S_j\rangle\langle S_j|$. By definition 3.2.2, the observable P_j takes value 1 if the system under measurement is found in the state $|S_j\rangle$ and zero in all orthogonal states. Therefore the mean value of this observable is exactly the probability to find the state $|S_j\rangle$ during measurement. Starting from the formula (6) we find

$$\langle P_j\rangle = \langle\Psi|S_j\rangle\langle S_j|\Psi\rangle = |\alpha_j|^2. \quad \textbf{(15)}$$

This is the projection postulate prediction to find the state $|S_j\rangle$ with the probability $|\alpha_i|^2$.

Now prove that the projection postulate implies mean value law (6). Consider any observable M with the spectrum of values $\{m_i\}$ and with complete set of eigenfunctions $|S_i\rangle$. Then the operator of the observable is represented by (13). Let $p(m_i)$ be probability to obtain value m_i in measurement. By definition 3.2.2, to find the value m_i is the same as to find the state $|S_i\rangle$, therefore, according to the projection postulate, $p(m_i) = |\alpha_i|^2$. Then we find

$$\langle M\rangle = \sum_i p(m_i)m_i = \sum_i |\alpha_i|^2 m_i = \langle\Psi|M|\Psi\rangle. \quad \textbf{(16)}$$

We have got the mean value law in the right hand side of (16) .

Since the projection postulate and the mean value law implicate each other, they are equivalent.

3.3 Implicit use of projection postulate in decoherence theory

Since equation (5) for the reduced density operator of the device is an implication of the mean value law (6), and since the mean value law and the reduction postulate are equivalent, then (5) is an implication of the reduction postulate as well. Moreover, formula (3) for density operator of classical statistical mixture is equivalent to the mean value law (6). Since both (3) and (5) are essentially used in solving the quantum measurement problem by the decoherence theory, then the solution of the quantum measurement problem proposed by the decoherence theory implicitly uses the projection postulate. Therefore it is impossible to say that decoherence theory solves the problem of measurement in a sense that it can eliminate and overcome the projection postulate. One can say only that the decoherence theory uses projection postulate by a less direct way and on the higher level than the Copenhagen interpretation does. Technically this provides great advantages, since it permits dynamical description and investigation of a measurement process. However, from the fundamental point of view the measurement problem remains unsolved: just as before, we have unitary reversible dynamics of Shrödinger equation and no-unitary and irreversible projection postulate.

4. Quantum states of the mind, many-world interpretation and decoherence theory

The analysis of secton 3 may be improved by explicitly including into consideration not only the device D but also quantum states of the observer's mind and states of the environment. The question is: Couldn't we describing the observer's perception, or mind, directly on quantum level, obtain explicitly selection of an alternative by the observer? Consider a system consisting of the following four parts: S — the measured microsystem; D—the measuring device; M —the mind of the observer; and E — the macroscopic environment (which may include an arbitrary large fragment of the Universe, but not the entire remaining Universe). We assume that the composite system $S\otimes D\otimes M\otimes E$ is isolated and performs unitary evolution in time during measurement.

Note that the first trouble with this approach is that E generally could not be isolated from the remaining Universe in any approximation. Quantum entanglement of E with remaining Universe is inevitable. Yet we could not directly include entire Universe into consideration because entire Universe is not a subject of unitary evolution in time: there is no external time for entire quantum Universe (DeWitt, 1967). So the isolated system $S\otimes D\otimes M\otimes E$ is highly idealized object. This is a kind of artificial “island universe”. Consider the problem of measurement in this (unrealistic) approximation.

Suppose as above that the system S was in a superposition state $\alpha|S_a\rangle+\beta|S_b\rangle$ before measurement. We can consider the process of measurement, suggesting that the evolution operator U describing the complete measurement acts as follows:

$$\begin{aligned} U|S_a\rangle|D_0\rangle|M_0\rangle|E_0\rangle &= |S_a\rangle|D_a\rangle|M_a\rangle|E_a\rangle; \\ U|S_b\rangle|D_0\rangle|M_0\rangle|E_0\rangle &= |S_b\rangle|D_b\rangle|M_b\rangle|E_b\rangle \end{aligned} \quad \textbf{(17)}$$

One should understand that the evolution (17) may be generally a multistep process. If we assume, for example, that S interacts directly only with the device D, the device interacts only with the observer's mind M, and the observer's mind interacts with the environment E then it would be three-step process: $|S_a\rangle$ implies $|D_a\rangle$, $|D_a\rangle$ implies $|M_a\rangle$, $|M_a\rangle$ implies $|E_a\rangle$ (and the same for b). But these details are not of high importance for the analysis (except that S does not interact with M directly—quantum system is beyond the horizon of perception of mind). Note that we consider consciousness as a set of states of the material system M which represents mind, or brain, or some substructures of brain of the observer. M_a means the state of consciousness when the observer perceives the result of measurement a, and analogously for b.

Linearity of the evolution operator U implies that the measurement on the system S in state $\alpha|S_a\rangle+\beta|S_b\rangle$ may be written as

$$U(\alpha|S_a\rangle+\beta|S_b\rangle)|D_o\rangle|M_o\rangle|E_o\rangle= \alpha|S_a\rangle|D_a\rangle|M_a\rangle|E_a\rangle+\beta|S_b\rangle|D_b\rangle|M_b\rangle|E_b\rangle =|\Psi_f\rangle. \quad \textbf{(18)}$$

It is seen that the state $|\Psi_f\rangle$ includes both possible final states of the mind $|M_a\rangle$ and $|M_b\rangle$ (and both states of the environment $|E_a\rangle$ and $|E_b\rangle$ as well). Our artificial island universe was splitted into two different branches corresponding to different possible outcomes of the measurement. In fact this is just what is supposed in the well known many-worlds approach to quantum mechanics interpretation (Everett, 1957; DeWitt and Graham, 1973). It is clearly seen that splitting of the universe in two branches is a direct and simple implication of linearity of the evolution operator U. Having such a splitting of the composite system one could not immediately say anything about the mechanism of selection of an alternative in the mind of the observer. To solve this problem many-worlds interpretation proclaims *selection postulate*: after the measurement the mind *finds itself* in only one of the branches of the superposition (18) while all branches still exist. Note that one more problem of the many-worlds approach (in the addition to the above mentioned problem with unitary evolution of the Universe) is that the meaning of the word 'exist' in this context remains completely unclear since this 'existence' could not be confirmed by observations (at least it cannot be confirmed without any additional hypothesis).

The selection postulate of many-worlds interpretation may be supported in the following way (Everett, 1957). Each component of $|\Psi_f\rangle$ is the state of the measured system (together with the device) corresponding to some definite result of the measurement and to the correlated with it definite state of the *memory* of the observer. We know from our experience that subjectively memory concerning any event can only be in one definite state. Therefore, each component of $|\Psi_f\rangle$ really *is* one definite subjective state of the observer after the measurement—one possible result of the measurement. This sounds reasonably but it is not a proof of the selection postulate, because the question remains: why we feel subjectively only one definite state of the memory while a number of states of the memory really exist in the final superposition? We can't answer this question, we can only fix that it is a matter of fact and a property of the mind. And it is just the content of the selection postulate.

To connect the mathematics of many-worlds interpretation with observations, many-worlds interpretation must have some quantitative rule to produce probabilities besides the above mentioned selection postulate. If the probability rule tells that the probability of mind to find itself in a branch is proportional to the square of norm of this branch in the superposition $|\Psi_f\rangle$, then operationally the selection postulate together with the probability rule is equivalent to the usual projection postulate and produces all usual results of quantum mechanics. The only change is that the words about reduction of the entangled state (or even initial state of the measured system) are replaced by the words about selection of a branch by the mind. *Effectively* we again have unitary dynamics plus unexplained projection postulate, but *heuristically* it is insisted in the many-worlds interpretation that no irreversible reduction of states occurs because all alternatives `coexist'.

The problem of selection of a branch in many-worlds approach like (18) may be considered also within decoherence theory. To understand the situation better one can ask about the state of the mind of the observer after the measurement, irrespective of all other subsystems. This is important because it is only the state of mind that is 'truly directly observable' by the observer (we have nothing for our experience but our impressions). To find out the answer, one should calculate the reduced density operator for the system M (mind) in the state $|\Psi_f\rangle$:

$$\rho_M = \sum_{i,j,k=a,b} \langle S_i | \langle D_j | \langle E_k | \Psi_f \rangle \langle \Psi_f | S_i \rangle | D_j \rangle | E_k \rangle = | M_a \rangle | \alpha |^2 \langle M_a | + | M_b \rangle | \beta |^2 \langle M_b |. \quad \textbf{(19)}$$

Equation (19) presents classical probability distribution on two possible states of mind $|M_a\rangle$ and $|M_b\rangle$. This may be interpreted as an evidence that the mind finds itself either in the state $|M_a\rangle$ or in the state $|M_b\rangle$ with probabilities $|\alpha|^2$ and $|\beta|^2$ respectively, but not in both states at the same time—like in classical probabilistic experiments (see discussion in section 2). We may seemingly conclude that the observer *subjectively* selects just one alternative (since we have *classical* distribution of probability for the states of the memory) although both alternatives actually coexist in the final state of the experiment, as it is seen from (18). However, we must remember that this conclusion depends on (19) and this equation depends on the projection postulate as was explained in section 3. In the present case projection postulate was applied on even higher level than in the usual decoherence theory (it is now applied to the composite system including not only quatum system and device, but also mind and environment). But the projection postulate already presupposes selection of an alternative and irreversible evolution, therefore we have again no solution to the measurement problem. Rather, this is a step back relative to original many-worlds interpretation.

5. Is projection postulate necessary indeed?

As we have seen in section 3, one needs to postulate (in some form or another) selection of an alternative even to have the right to talk about observations. Yet there is no way to connect theory with reality without observables or observations. Therefore some selection must be postulated explicitly in the theory. But the situation with quantitative or probabilistic content of measurement postulates is different. Actually, the explicit formula for the Born's rule for the probabilities may be deduced from other postulates that may look more simple, more fundamental or more natural. There are a number of approaches of this type (see for references (Zurek, 2007b), but we consider here explicitly only three ones.

5.1 Approach of H. Everett III

The first approach is the one of Hugh Everett III—the author of many-worlds interpretation. The main aim of his seminal paper (Everett, 1957) was to reformulate or to generalize quantum mechanics to make it applicable to such fundamental structures as the space-time itself. The main problem he saw on this way is splitting of quantum mechanics on unitary and reversible quantum dynamics and no-unitary and irreversible projection postulate. Everett proclaimed that his goal was to propose such a formulation of quantum mechanics which includes no projection postulate but which can nevertheless be deduced from the conventional formulation. So efforts were applied in (Everett, 1957) to deduce projection postulate from its new 'relative-state' formulation (later frequently known as many-worlds interpretation). Everett understood that some selection postulate is inevitable to connect theory with reality, and he explicitly formulated that "Each branch represents a different outcome of the measurement" (Everett, 1957), p.320. This statement cannot be deduced from anything other. Besides, he tried to derive quantitative rules for probabilities. His method is the following.

Let $|\phi_i\rangle$ be a set of orthonormal states of some system (may be composed) and consider a superposition

$$\sum_i a_i |\phi_i\rangle = \alpha |\phi'\rangle, \quad \textbf{(20)}$$

where the state $|\phi'\rangle$ is supposed to be normed to 1. The question is, what is the probability measure of the term $\alpha|\phi'\rangle$ if included in some other superposition. Everett formulates two postulates to deduce this probabilistic measure.

Postulate 5.1.1 *The probability measure of $\alpha|\phi'\rangle$ depends only on $|\alpha|$ and does not depend on the phase of α and on the nature of the state $|\phi'\rangle$.*

It is followed from the postulate 5.1.1[2] that the probability measure for $\alpha|\phi'\rangle$ may be written as $m(|\alpha|)$.

Postulate 5.1.2 (Additivity) *For the term* $\alpha|\phi'\rangle$ *defined by (20) it holds* $m(|\alpha|)=\sum_i m(|a_i|)$.

The probability rule of the projection postulate is easily deduced from the above postulates. Actually, from (20) we have

$$|\alpha|=\left|\sum_i a_i|\phi_i\rangle\right|=\sqrt{\sum_i|a_i|^2}. \qquad \textbf{(21)}$$

Then from the postulate 5.1.2 and from (21) we have

$$m\left(\sqrt{\sum_i|a_i|^2}\right)=\sum_i m\left(\sqrt{|a_i|^2}\right). \qquad \textbf{(22)}$$

It follows immediately from (22) that $m(\sqrt{x})=cx$ for any $x\geq 0$. Here c is some constant. Finally we obtain

$$m(|\alpha|)=c|\alpha|^2. \qquad \textbf{(23)}$$

Up to a constant factor c equation (23) coincides with the probability rule of the projection postulate. The value of the constant c is fixed as $c=1$ by condition that the measure of a single normalized state is equal to 1. To summaries, Everett's approach is based on additivity of probabilistic measure of orthogonal states. Note that the idea of the Everett's proof is somewhat similar to the Gleason's theorem (Gleason, 1957).

5.2 Approach of D. Deutsch

The second approach is due to paper of David Deutsch (Deutsh, 1999). We do not reproduce exactly the Deutsch's arguments since in our opinion they are too sophisticated (based on the games theory, decision theory and related notions) and look to be not quite precise in some points (we mention two items below). However, the idea is clear and interesting. We provide below a simplified argumentation stimulated by the Deutsch's paper (Deutsh, 1999) but not coinciding with the arguments given in this paper.

[2]Actually Evetrett formulated this condition but did not give it the status of a postulate.

Our argumentation will be related to the mean value law (6) rather than the probabilistic measure of states like in the above Everett's argumentation. But we already have seen that the mean value law is equivalent to the projection postulate (see section 3.2).

Consider a quantum system χ with n-dimensional space of states. We use the definitions of measurement and observable 3.2.1 and 3.2.2. Let X be an observable and $|1\rangle,\ldots,|n\rangle$ be a basis of corresponding normalized eigenstates, therefore the operator of the observable is $X=\sum_i x_i|i\rangle\langle i|$, where x_i are the values of the observable that it takes in single measurements.

Postulate 5.2.1 *For any state* $|\Psi\rangle$ *of the system* χ *and for a measurement related to any orthonormal basis* $|1\rangle,\ldots,|n\rangle$, *for the measurement in the initial state* $|\Psi\rangle$ *there are probabilities* p_i *to find each state* $|i\rangle$ *of the basis.*

Corollary 5.2.1 *For any observable* X *with eigenbasis* $|1\rangle,\ldots,|n\rangle$ *and with spectrum* $x_1,\ldots,x_n$ *there exists a mean value* $\langle X\rangle$ *for the measurement of the system in the state* $|\Psi\rangle$ *and*

$$\langle X\rangle=\sum_i p_i x_i. \qquad \textbf{(24)}$$

Proof. Since $x_1,\ldots,x_n$ is only a scale of the measurement, then the probability of appearance of each value x_i due to postulate 5.2.1 is p_i and by the meaning of probability we have $\langle X\rangle\equiv\langle x\rangle=\sum_i p_i x_i$. ■

The mean value $\langle X\rangle$ for the observable with basis $|i\rangle$ and spectrum x_i measured in the state of the system $|\Psi\rangle$ may be expressed in two equivalent forms:

$$\langle X\rangle=\nu\left(\sum_i x_i|i\rangle\langle i|;|\Psi\rangle\right)\equiv\nu(x_1,\ldots,x_n;|\Psi\rangle). \qquad \textbf{(25)}$$

Our purpose is to find the function $\nu(x_1,\ldots,x_n;|\Psi\rangle)$. We do not suppose the state $|\Psi\rangle$ to be normalized to unity, that is we try to prove the generalized form of the mean value law for no-normalized states:

$$\langle X\rangle=\langle\Psi|X|\Psi\rangle/\|\Psi\|^2. \quad \textbf{(26)}$$

To prove (26) we need further postulates.

Postulate 5.2.2 *Let* $|\Psi\rangle=\sum_i a_i|i\rangle$ *. Then* $|a_i|=|a_j|$ *implies* $p_i=p_j$ *.*

Note that postulate 2.2 was not explicitly formulated in (Deutsh, 1999) but some similar statement was implicitly used in transition from eq. (10) to eq. (11) in (Deutsh, 1999).

Postulate 5.2.3 *Let* $|\Psi\rangle=\sum_i a_i|i\rangle$ *. Then* $a_i=0$ *implies* $|p_i|=0$ *.*

Corollary 5.2.2 *If for some* $m\le n$

$$|\Psi\rangle=\sum_{j=1}^{m}|i_j\rangle$$

and $X=\sum_i x_i|i\rangle\langle i|$, then

$$\langle X\rangle=\frac{1}{m}\sum_{j=1}^{m}x_{i_j}. \quad \textbf{(27)}$$

Proof. Formula (27) is a direct implication of equation (24) and postulates 5.2.2 and 5.2.3. ■

Now consider a system with two-dimensional space of states and prove that for any integer numbers k and l the following equality holds:

$$v(x_1,x_2;\sqrt{k}|1\rangle+\sqrt{l}|2\rangle)=\frac{kx_1+lx_2}{k+l}. \quad \textbf{(28)}$$

Equation (28) is in fact all that we need to prove the mean value law (26) in the general case. First, equation (28) is a particular case of (26) which under appropriate normalization of the state may be extended to any rational coefficients at $|1\rangle$ and $|2\rangle$ in two-dimensional state space. Further, from the physical point of view there is no difference between rational and any real coefficients since any real number may be approximated by rational numbers with any desired precision. If the coefficients are any complex numbers then the related phases may be accounted for by redefinition of the states $|1\rangle$ and $|2\rangle$. Thus we have got mean value law (26) for a generic two-dimensional case. Finally, the proof that is represented below may be extended to any n-dimensional case in a straightforward way.

To prove (28) we need additional postulate:

Postulate 5.2.4 *Let* χ *be an observable of system* χ *with orthonormal eigenbasis* $|1\rangle,\ldots,|n\rangle$ *and let* $|Y_1\rangle,\ldots,|Y_n\rangle$ *be any orthonormal states of some other quantum system* Y *. Then for any* $a_1,\ldots,a_n$

$$v(x_1,\ldots,x_n;\sum_i a_i|i\rangle)=$$
$$v(x_1,\ldots,x_n;\sum_i a_i|i\rangle|Y_i\rangle).$$

Note that it is very fine postulate. This postulate supposes that in the left hand side the mean value $\langle X\rangle$ is measured in a pure state of the system χ but in the right hand side there is no pure state for χ. It is supposed that if one entangles the system χ with other system $\mathcal{Y}$ then 1) measurement on the observable X is still possible and 2) the mean value will be the same as for the isolated system. Note that this (or similar) postulate was not explicitly formulated in the paper (Deutsh, 1999) while somewhat like this was implicitly used in the transition from equation (12) to equation (21) of the paper (Deutsh, 1999).

Now let us prove (28) . Suppose $\mathcal{Y}$ to be $k+l$ dimensional quantum system with orthonormal basis $|y_1\rangle,\ldots,|y_{k+l}\rangle$ and let

$$|Y_a\rangle=\frac{1}{\sqrt{k}}\sum_{i=1}^{k}|y_i\rangle;\quad |Y_b\rangle=\frac{1}{\sqrt{l}}\sum_{i=k+1}^{k+l}|y_i\rangle. \quad \textbf{(29)}$$

From postulate 2.4 we have:

$$v(x_1,x_2;\sqrt{k}|1\rangle+\sqrt{l}|2\rangle)=$$
$$v(x_1,x_2;\sqrt{k}|1\rangle|Y_a\rangle+\sqrt{l}|2\rangle|Y_b\rangle). \quad \textbf{(30)}$$

On the other hand, to measure X in composite system $\mathcal{X}\otimes\mathcal{Y}$ (as prescribed by right hand side of (30)) means to measure the observable $X_{XY}=X\otimes I_{\mathcal{Y}}$ on this system. Actually, the states $|i\rangle|y_j\rangle$ form an eigenbasis of the $X_{\mathcal{XY}}$, each time when $i=1$ the result of measurement is x_1, and each time when $i=2$ the result of measurement is x_2. The mean value of $X\otimes I_{\mathcal{Y}}$

in the entangled state $\sqrt{k}|1\rangle|Y_a\rangle+\sqrt{l}|2\rangle|Y_b\rangle$ may be written as

$$\begin{aligned}&\langle X_{\mathcal{XY}}\rangle = \\ &\nu\Bigg(x_1\sum_{i=1}^{k+l}|1\rangle|y_i\rangle\langle 1|\langle y_i| + x_2\sum_{i=1}^{k+l}|2\rangle|y_i\rangle\langle 2|\langle y_i|; \\ &\quad 1\times\sum_{i=1}^{k}|1\rangle|y_i\rangle + 0\times\sum_{i=k+1}^{k+l}|1\rangle|y_i\rangle + \\ &\quad 0\times\sum_{i=1}^{k}|1\rangle|y_i\rangle + 1\times\sum_{i=k+1}^{k+l}|1\rangle|y_i\rangle\Bigg).\end{aligned} \qquad (\mathbf{31})$$

It is seen from (31) that the mean value $\langle X_{\mathcal{XY}}\rangle$ is measured for the state which is a superposition of a number of eigenstates with the same weights 1. Therefore it follows from equation (27) that the mean value must be an averaged value of corresponding eigenvalues. Since there are totally $k+l$ items in this superposition, k items correspond to the eigenvalue x_1 of the observable $X_{\mathcal{XY}}$, and l items correspond to the eigenvalue x_2, we immediately obtain from (31) :

$$\nu(x_1,x_2;\sqrt{k}|1\rangle+\sqrt{l}|2\rangle)=\langle X_{XY}\rangle=\frac{kx_1+lx_2}{k+l}. \qquad (\mathbf{32})$$

which had to be proved.

5.3 Approach of W. H. Zurek

Finally we would like to mention an approach proposed by W. H. Zurek as this approach is deep and very instructive. We will refer the most recent paper (Zurek, 2007b) but actually this approach was developed by Zurek in a number of papers (see (Zurek, 2007a; 2005) and references in (Zurek, 2007b)).

Zurek emphasizes that usual approach to decoherence theory which based on using of reduced dencity matrix implicitly uses projection postulate and produces logical circles. Since projection postulate is in the deep contradiction with the main unitary part of quantum theory, then one of the main goal is to remove projection postulate from the theory in *both* its aspects: selection part and probabilistic part. He proclaims (Zurek, 2007b), p. 4: “instead of the demand of a single outcome we shall only require that the results of the measurement can be confirmed (by a re-measurement), or communicated (by making a copy of the record)”. To proceed Zurek starts with the following set of axioms (Zurek, 2007b), p.1, p.3, and statements which he calls 'Facts' (Zurek, 2007b), p.11 (we use notations of (Zurek, 2007b)):

(o) *The Universe consists of systems.*

(i) *State of a quantum system is represented by a vector in its Hilbert space* $\mathcal{H}_s$.

(ii)*Evolutions are unitary (i.e., generated by Schrödinger equation).*

(iii) *Immediate repetition of a measurement yields the same outcome.*

Fact 1: Unitary transformations must act on the system to alter its state. That is, when an operator does not act on the Hilbert space $\mathcal{H}_s$ of $\mathcal{S}$, i.e., when it has a form $\ldots\otimes\mathbf{1}_{\mathcal{S}}\otimes\ldots$ the state of $\mathcal{S}$ does not change.

Fact 2: Given the measured observable, the state of the system $\mathcal{S}$ is all that is needed (and all that is available) to predict measurement results, including probabilities of outcomes.

Fact 3: The state of a larger composite system that includes $\mathcal{S}$ as a subsystem is all that is needed (and all that is available) to determine the state of $\mathcal{S}$.

Axioms (o)-(ii) declare standard unitary part of quantum mechanics and the axiom (iii) introduces the notion of measurement. It is seen from axiom (iii) that measurement is something that could be carried out or applied many times and that a measurement has some outcome. But it is not declared explicitly *what is* this outcome and this is in agreement with the tendency to avoid an explicit declaration of selection of an alternative. Then Zurek considers a system $\mathcal{S}$ interacting with another quantum system $\mathcal{E}$ (apparatus or environment) and he writes literally the following (Zurek, 2007b), p.4:

Let us suppose (in accord with axiom (iii)) that there is a set of states which remain unperturbed by this interaction - *e.g.*, that this interaction implements a measurement - like information transfer from $\mathcal{S}$ to $\mathcal{E}$: $|s_k\rangle|\varepsilon_o\rangle \Rightarrow |s_k\rangle|\varepsilon_k\rangle$.

Then he considers outcomes of a measurement exclusively as some pure states. However this is nothing else than an implicit postulating that the outcome of a measurement *is* a single pure state $|s_k\rangle$ of the measured system $\mathcal{S}$, that is to say a measurement *creates* a pure state of a measured system. In the opposite case we could not apply to such outcome axiom (iii). And this is just a kind of a postulate of selection of an alternative. This of course *is* the demand of a single outcome of a measurement in the contradiction with the declared purposes in (Zurek, 2007b), p.4, as was cited above. A selection postulate is unavoidable again.

One should note that the selection postulate implicitly used by Zurek is weaker than one in our definition of measurement 3.2.1 since the complete set of the possible outcomes is not fixed yet. May be, it is one of the *weakest* possible form of the selection postulate---this item should be studied (possible generalization is a subspace of Hilbert space as a possible outcome of the experiment).

Then, instead of postulating of orthogonality of the possible outcomes of the measurement, with equation $|s_k\rangle|\varepsilon_o\rangle \Rightarrow |s_k\rangle|\varepsilon_k\rangle$ and with axioms (i) and (ii) Zurek proves that all possible outcomes $|s_k\rangle$ *must* be orthogonal. Further, with use of very useful and deep notion of *envariance* (entanglement - assisted invariance) (Zurek, 2007b), p.9, and Facts 1-3 he proves some statement that is equivalent to our postulates 5.2.2 and 5.2.3. That is, in the approach of Zurek (Zurek, 2007b) postulates 5.2.2 and 5.2.3 and orthogonality of outcomes are theorems. The rest of the proof that related to the Born's probability rule is similar to one of D. Deutch (Deutsch, 1999) and presented above simplification of his proof (section 5.2).

Thus we see that while some sort of a postulate about selection of an alternative absolutely cannot be avoided if we would like to have a notion of observable or observation, the exact probability rules for this selection and complete set of outcomes must not be postulated. Rather all these may be deduced from other axioms that may look simpler or more suitable from different points of view. But we would like to stress that all these kinds of axioms while may look natural or simple, lay beyond the framework of the quantum linear dynamics and are *arbitrary* in respect to the quantum dynamical laws. The same is true for usual projection postulate of course. Therefore a freedom in selection of probability laws or related postulates is seemingly restricted only by experiment and is not restricted by the linear dynamical part of quantum mechanics. One can suppose that having the same linear and unitary quantum mechanics it is possible in principle to obtain quite different no-trivial physics with using different sets of additional interpretation (measurement) axioms. But is it possible really, even in some logically complete toy model?

6. Discussion

To conclude, the decoherence theory and many-worlds approach are large steps toward understanding of quantum measurement problem, however they do not provide a complete and fundamental solution of this problem. The reason is that these theories could not completely eliminate selection postulate and probabilistic rules or postulates that replace probabilistic rules. The latters lay outside of the linear quantum dynamics. Selection postulates and probability rules look to be clearly separated from linear dynamics of quantum theory and look to be somewhat obscure things. This is why a large field exists for further discussion of them as well as for speculations and unorthodox approaches like extended Everett's concept or postcorrection (Mensky, 2007). These capabilities should be thoroughly studied. The only thing that may point at the connection between quantum dynamics and additional postulates is that the quantum dynamics is not only linear but it is unitary. Such dynamics conserves normalization of state vectors that may point out on some important role of the normalization. But this role cannot be clarified only within the quantum dynamics.

One can note that the positions of the decoherence theory and many-worlds approach in this respect are somewhat different. While the decoherence theory could describe a measurement purely dynamically, in its interpretation part the notion of density operator is essentially used in different contexts. This is directly connected with the projection postulate. Therefore, the decoherence theory still supposes splitting of dynamics in two parts. Note also that the usage of other sets of axioms to deduce probability law instead of explicit mean value law (6) or explicit Born's rule $p_i = |a_i|^2$ of the projection postulate (as was described in section 5) does not eliminate irreversible dynamics during measurement on some stage of the interpretation.

At the same time many-worlds interpretation postulates *finding of an observer himself*, with some probability, within one of the branches of the quantum Universe while all the branches still coexist. Many-worlds interpretation says nothing about evolution which is irreversible and not unitary. In this respect this theory is in better logical agreement with unitary dynamics than the decoherence theory. But (as was pointed out above, section 4) the notion of coexisting different branches of the quantum Universe remains meaningless from the operational point of view. Moreover, logically closed many-worlds interpretation must include quantum cosmology without unitary evolution in external time. Rather, one should expect that time in this context would be an emergent phenomenon.

Finally, some kind of the selection postulate is an absolutely necessary part of any interpretation of, or approach to linear quantum theory. And any kind of selection postulate lays outside of linear quantum theory itself.

In fact, need of a selection postulate is merely a consequence of the way in which human mind or consciousness perceives the nature and may be considered as the property of the mind. We cannot (in clear mind at any rate) split our consciousness in a number of branches with a number of amplitudes and to think by such a "quantum logic". It is a consequence of classical nature of the mind, and need in a selection postulate looks as a consequence of the same thing.

Consciousness fixes the experience of perception of the nature and treats this experience as an informational images, and information is also a classical thing. Information supposes some kind of fixation of its content on some classical medium. The main way to think clearly and carefully about something in physics or about any other field is to process information by a logic—to think mathematically. Actually this concerns not only information about real nature. All branches of mathematics including "pure mathematics" presuppose possibility to fix information—axioms, theorems, proofs—on a classical medium (or, equivalently, they may be realized in operation of some classical apparatus like Turing's machine). Each mathematical proof or calculation must allow its realization as a physical process, at least in principle. There is no mathematics without possibility to fix information in a classical way. Consequently existence of mathematics is an implication of existence of classical world or, more exact, classical branches of the quantum Universe (in terms of many-worlds interpretation). Mathematics as a whole, including pure mathematics, is not a "clear mind product", rather it has definite physical background, it cannot exist without classical world and therefore it is connected with decoherence theory, problem of selection, and again with the nature of the mind, and so on.

From this point of view one may note that there exist obvious possible generalization of the notion of mathematics. This generalized notion may be called "partially quantum mathematics" and it supposes that some parts of a proof or a calculation are carried out without fixation of each intermediate steps in a classical medium, but by a pure quatum way with use of linear superposition of states of some agent. But some stages of calculation or at least the initial and the final states of the calculation must be classical—in opposite case there may be no connection of such a proof or calculation with our mind. Also some "rules" of the quantum proof must be written out in some classical medium as a kind of program to drive the quantum agent ("quantum computer"). No "pure quantum mathematics", without fixation the results and quantum rules in a classical medium, can exist. In the opposite case we would need a ``quantum mind'' to operate such pure quantum mathematics.

These arguments show that the selection postulate in quantum theory is a very fundamental entity tightly connected with existence of mathematics and with the nature of the mind—all these concepts are of the same depth. Selection postulate in some form or another is a necessary precondition even to start development of a scientific description of nature on quantum level. But the probability rules look more technical and admit various approaches on the base of various sets of axioms.

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